Boost invariant quantum evolution of a meson field at large proper times

Dominique Vautherin
Division de Physique Théorique *
Institut de Physique Nucléaire,
F-91406 , Orsay Cedex, France

and

T. Matsui
Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606, Japan

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Abstract

We construct asymptotic solutions of the functional Schrödinger equation for a scalar field in the Gaussian approximation at large proper time. These solutions describe the late proper time stages of the expansion of a meson gas with boost invariant boundary conditions. The relevance of these solutions for the formation of a disoriented chiral condensate in ultra relativistic collisions is discussed.

*Unité de Recherche des Universités Paris XI et Paris VI associée au C.N.R.S

Introduction. — Several recent studies have been devoted to the calculation of the evolution of a self interacting scalar field. As an example the evolution of a scalar field is an important ingredient in most inflationary models [1, 2, 3, 4, 5]. Also classical as well as quantum calculations of the evolution of a scalar field have been considered in the context of the possible formation of a disoriented chiral condensate in ultrarelativistic nuclear collisions [6, 7, 8, 9, 10].

In this note we investigate the asymptotic forms of the quasi-one-dimensional quantum evolution of the meson fields which possess a symmetry with respect to the Lorentz boost along z-direction. Such a symmetry was originally introduced by Bjorken[11] in a hydrodynamical model of ultrarelativistic nucleus-nucleus collision based on the physical assumption of central plateau formation in the final particle distribution in the rapidity space.

We take a simple model for mesons described by a self-interacting scalar field $\varphi(\mathbf{x})$ and adopt the functional Schrödinger picture with the Hamiltonian density:

$$\mathcal{H}(\mathbf{x}) = \frac{1}{2}\pi^2(\mathbf{x}) + \frac{1}{2}(\vec{\nabla}\varphi(\mathbf{x}))^2 + \frac{m_0^2}{2}\varphi^2(\mathbf{x}) + \frac{\lambda}{24}\varphi^4(\mathbf{x}) \ .$$

We use the framework of the Gaussian or mean field approximation, i.e. we assume that at each time t the wave functional describing the system is a gaussian type coherent state

$$\begin{aligned} \Psi\left[\varphi(\mathbf{x})\right] &= \mathcal{N} \exp\left(i < \pi|\varphi - \bar{\varphi} >\right) \\ &\quad \times \exp\left(- < \varphi - \bar{\varphi}|(\frac{1}{4G} + i\Sigma)|\varphi - \bar{\varphi} > \right), \end{aligned}$$

where G, Σ, $\bar{\varphi}$, π define respectively the real and imaginary part of the kernel of the Gaussian and its average position and momentum. We impose Lorentz invariant boundary conditions i.e. we assume that observables depend only on the proper time variable $\tau = \sqrt{t^2 - z^2}$. We examine the behaviour of the meson wave functional for large values of the proper time variable. We will show that this limit can be worked out analytically by solving the linearized evolution equations.

For more simplicity we restrict ourselves to the case of a single scalar field with an initial condition which reduces to a single pure state (not a statistical mixture). As will be shown below this case is already instructive

and general enough to provide a transparent interpretation of some of the numerical calculations performed recently by Cooper *et al* [12] in the more general case of a sigma model with an initial state which is a statistical mixture. A more specific and more detailed comparison adapted to this case will be presented in a forthcoming publication.

Mean Field Equations. — The evolution equations in the Gaussian approximation read [2, 4, 5]

$$\begin{aligned}
\dot{G} &= 2(G\Sigma + \Sigma G), \\
\dot{\Sigma} &= \frac{1}{8}G^{-2} - 2\Sigma^2 - \frac{1}{2}\{-\Delta + m_0^2 + \frac{\lambda}{2}\bar{\varphi}^2 + \frac{\lambda}{2}G(\mathbf{x},\mathbf{x})\} \\
\dot{\bar{\varphi}} &= -\bar{\pi}, \\
\dot{\bar{\pi}} &= \{-\Delta + m_0^2 + \frac{\lambda}{6}\bar{\varphi}^2 + \frac{\lambda}{2}G(\mathbf{x},\mathbf{x})\}\bar{\varphi}.
\end{aligned}$$

In these equations m_0 is the bare mass of the scalar field and λ the bare strength of the self coupling term which we write as $\lambda\varphi^4$ in the Lagrangian density. The vacuum state corresponds to the static solution

$$\Sigma = 0, \quad \bar{\pi} = 0, \quad \bar{\varphi} = \varphi_0, \quad G = 1/2\sqrt{-\Delta + \mu^2},$$

where the quantity μ has to satisfy the so- called gap equation [13]

$$\mu^2 = m_0^2 + \frac{\lambda}{2} < \mathbf{x}|\frac{1}{2\sqrt{-\Delta + \mu^2}}|\mathbf{x} > + \frac{\lambda}{2}\varphi_0^2. \tag{1}$$

The previous equation requires a regularization scheme such as a discretization of the Laplacian operator on a lattice with a mesh size $\Delta x = 1/\Lambda$ or a cutoff Λ in momentum space. To make the gap equation finite when the scale Λ goes to ∞ a popular prescription is to send the bare coupling constant to zero according to the formula [14]

$$\frac{1}{2\lambda_R} = \frac{1}{\lambda} + \frac{1}{16\pi^2}\log(\frac{2\Lambda}{e\mu}).$$

This prescription is however not free from difficulties, see [3, 13, 14].

The mean field evolution equations can be written in a more compact form which has the further advantage of being manifestly covariant

$$
\begin{aligned}
m^{2}(x) &= m_0^2 + \frac{\lambda}{2}\bar{\varphi}^2(x) + \frac{\lambda}{2} < x|S|x > , \\
\{ \quad \Box + m_0^2 &+ \frac{\lambda}{6}\bar{\varphi}^2 + \frac{\lambda}{2} < x|S|x > \}\bar{\varphi} = 0,
\end{aligned}
$$

where $x = (x_0, x_1, x_2, x_3)$ and where S is the Feynman propagator in the presence of an x dependent mass

$$
S = \frac{i}{\Box + m^2(x) + i\varepsilon}.
$$

A proof of this equation can be constructed by using the functional methods of Cornwall, Jackiw and Tomboulis [15]. The kernel G of the Gaussian wave functional is related to the propagator S via the relation

$$
< \mathbf{x}|G(t)|\mathbf{x} > = < x|S|x > .
$$

To obtain the asymptotic form of the state of the meson field at large proper times we assume that the propagator S is close to its vacuum value. In this case we linerize the equations of motion around the vacuum value by writing

$$
S = \frac{i}{\Box + \mu^2 + \delta m^2(x) + i\varepsilon} = S_0 + S_0 \ (\delta m^2) \ S_0 + \ldots \tag{2}
$$

Introducing the Fourier decomposition

$$
\begin{aligned}
\delta \quad m^2(x) &= \int \frac{d^4q}{(2\pi)^4} \exp(iq\cdot x)\delta m^2(q), \\
\delta \quad \varphi(x) &= \bar{\varphi}(x) - \varphi_0 = \int \frac{d^4q}{(2\pi)^4} \exp(iq\cdot x)\delta\varphi(q),
\end{aligned}
$$

equation (2) leads to

$$
< x|\delta S|x > = \int \frac{d^4q}{(2\pi)^4} \exp(iq\cdot x)\Pi_0(q^2)\delta m^2(q),
$$

where $\Pi_0(q^2)$ is the standard polarization tensor

$$\begin{aligned}\Pi_0(q^2) &= -\int \frac{d^4p}{(2\pi)^4} S_0(p) S_0(p+q) \\ &\simeq -\frac{1}{16\pi^2}\{\log(\frac{4\Lambda^2}{e^2\mu^2}) + \frac{q^2}{\mu^2} + \ldots\}.\end{aligned}$$

The linearized equations of motion are thus found to be

$$\begin{aligned}\delta \quad & m^2(q) = \lambda\varphi_0\delta\bar{\varphi}(q) + \frac{\lambda}{2}\Pi_0(q^2)\delta m^2(q), \\ (\quad & -q^2 + \mu^2)\delta\bar{\varphi}(q) + \frac{\lambda}{2}\varphi_0\Pi_0(q^2)\delta m^2(q) = 0.\end{aligned}$$

These equations are equivalent to those derived by Kerman and Lin [14] in the functional Schrödinger picture. A compact and elegant construction is provided by our manifestly covariant formalism.

When the momentum cutoff Λ goes to ∞ the quantity $\lambda\ \Pi_0$ has a finite limit while the product $\lambda\ \varphi_0$ goes to zero and the first equation becomes

$$\left[1 - \frac{\lambda}{2}\Pi_0(q^2)\right]\delta m^2(q) = 0.$$

The linearized evolution equations have two types of solutions. The first type of solutions are given by

$$\delta m^2 = 0, \qquad \delta\varphi(q) = f(q)\delta(q^2 - \mu^2),$$

where μ is the self consistent solution of the gap equation (1) and $f(q)$ is a regular function of $q = (q_0, q_1, q_2, q_3)$ to be specified later by boundary conditions. In the second type

$$\delta m^2(q) = \ (q^2 - \mu^2)\delta\varphi \qquad \delta\varphi = f(q)\delta(q^2 - M^2),$$

One important difference with the solutions of the first type is that now M^2 is a solution of the equation

$$1 - \frac{\lambda}{2}\Pi_0(M^2) = 0.$$

This equation is just the lowest order Bethe-Salpeter equation for the φ^4 field theory. For small enough values of the renormalized coupling it has a single solution M^2 i.e. a single bound state [14].

To construct Lorentz boost invariant solutions inside the forward light cone ($x_0^2 - x_3^2 = t^2 - z^2 > 0, \quad x_0 = t > 0$), it is convenient to introduce the light cone coordinates by

$$\begin{aligned} x_0 &= \tau \cosh \eta, \qquad x_3 = \tau \sinh \eta, , \\ q_0 &= \sigma \cosh y, \qquad q_3 = \sigma \sinh y, \end{aligned}$$

where $\sigma = \pm\sqrt{q_0^2 - q_3^2}$ and y is the usual rapidity variable. Then the Fourier integral for $\delta\varphi$ for the first type of solutions is written as

$$\begin{aligned} \delta\varphi(x) &= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\sigma \sigma \int_{-\infty}^{\infty} dy \int d\mathbf{q}_\perp f(q) \delta(q^2 - \mu^2) \\ &\quad \times \exp\left[i\sigma\tau \cosh(y - \eta) - i\mathbf{q}_\perp \cdot \mathbf{x}_\perp\right], \end{aligned}$$

where $\mathbf{x}_\perp = (x_1, x_2)$ and $\mathbf{q}_\perp = (q_1, q_2)$. We seek the form of $f(q)$ which gives $\varphi(x)$ independent of η and $\mathbf{x}_\perp$. By noting that $q^2 = \sigma^2 - \mathbf{q}_\perp^2$ does not depend on y, one can show by inspection that the required form is given by

$$f(q) = \delta^{(2)}(\mathbf{q}_\perp) g(\sigma)$$

Indeed, the Fourier integral can now be performed explicitly and the result can be written by a linear combination of the Bessel function and the Neumann function of order zero, J_0 and N_0:

$$\bar{\varphi}(x) = \varphi_0 + \alpha J_0(\mu\tau) + \beta N_0(\mu\tau),$$

where we have used the following integral expressions for J_0 and N_0 [16]:

$$\begin{aligned} J_0(x) &= \frac{2}{\pi} \int_0^{\infty} dt \sin(x \cosh t) \\ N_0(x) &= -\frac{2}{\pi} \int_0^{\infty} dt \cos(x \cosh t) \end{aligned}$$

The two coefficients α and β are given by the value of the function $g(\sigma)$ at $\sigma = \pm\mu$

$$\alpha = \frac{1}{32\pi^3}\left[g(-\mu) - g(\mu)\right], \quad \beta = \frac{i}{32\pi^3}\left[g(\mu) + g(-\mu)\right]$$

These two coefficients may be alternatively given by the initial conditions of $\varphi(\tau)$ and its derivative at $\tau = \tau_0$ when the evolution of the meson field starts.

The appearance of the Bessel (and Neumann) function may be easily understood if we rewrite the Klein-Gordon equation $(\Box + \mu^2)\,\varphi(x) = 0$ using the light cone variables [12]:

$$\left[\frac{\partial^2}{\partial\tau^2} + \frac{1}{\tau}\frac{\partial}{\partial\tau} - \frac{1}{\tau^2}\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial x_\perp^2} + \mu^2\right]\varphi(x) = 0.$$

For $\varphi(x)$ independent of η and $\mathbf{x}_\perp$, this equation becomes a form of Bessel's differential equations whose two independent (real) solutions are given by J_0 and N_0.

Boost invariant solutions of the second type can be obtained in a similar way:

$$\begin{aligned}\bar{\varphi}\quad(x) &= \bar{\varphi}_0 + \alpha J_0(M\tau) + \beta N_0(M\tau),\\ m\quad{}^2(x) &= \mu^2 + (M^2 - \mu^2)\left[\alpha J_0(M\tau) + \beta N_0(M\tau)\right].\end{aligned}$$

We thus learn that for large values of the proper time the system returns, as postulated above, to the vacuum state for both types of solutions, which justifies a posteriori the approximation scheme we have developped.

Discussion. — The first type of solution leads to a momentum distribution of the form $1/\sqrt{k^2+\mu^2}$ which is time independent and boost invariant. It describes the Lorentz invariant expanding coherent state of a meson gas. The second type of solution involves the solution of the lowest order Bethe-Salpeter equation and can thus be interpreted as an expanding coherent state of two-meson bound states. Solutions exist only for a negative bare coupling constant, as discussed in reference [14].

Although our formulae were derived for the specific case of a single scalar field with pure Gaussian states and a particular renormalization scheme involving a negative bare coupling, they exhibit general features of boost invariant boundary conditions which are expected to hold for other more general models. In this respect it is instructive to compare our formulae with the results obtained numerically by Cooper *et al* [12]. These calculations concern the case of a sigma model with statistical mixtures as initial conditions. They are expected to be relevant to discuss the possible formation of a chiral

condensate in ultra relativistic collisions. They use a *finite* momentum cutoff and a *positive* bare coupling constant, a case to which our formulae are not a priori immediately applicable. There is nevertheless a striking agreement between our formulae involving Bessel functions of order zero and figure 5 of Cooper *et al* which shows the variation of the square mass $\mu^2 + \delta m^2$ as a function of the proper time τ. It is interesting to note that the results of Cooper *et al* are well described by a *single* Bessel function whereas our previous analysis suggests a superposition of two Bessel functions with characteristic scales μ and M. One should remember however that since Cooper *et al* have a positive coupling constant there are no two- meson bound states. This explains why the scale M is absent but still leaves one unanswered question. Indeed without bound states our formulae predict no variation at all in the square mass. The answer to this question is that our derivation was made in the case of an infinite momentum cutoff whereas Cooper *et al* use a finite one. This difference produces the non vanishing value of δm^2.

By looking at the results of Cooper *et al* it is also worthwhile to note that some of the initial conditions make the small amplitude approximation valid for the whole evolution. Our formulae also explain nicely why increasing the formation time leads to larger asymptotic oscillations as a result of the $1/\sqrt{\tau}$ behavior of the Bessel functions. They do not explain why there are in this case less instabilities since this involves the small proper time domain in which our approximation scheme is expected to break down. Our analytic asymptotic expressions appear nevertheless as a useful reference to analyze some of the physics of the boost invariant expansion of a meson gas. Results using the sigma model with a finite cutoff will be presented in a forthcoming publication.

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References

[1] A. Guth and S. Y. Pi, Phys. Rev. **D32**, 1899 (1985).

[2] O. Eboli, R. Jackiw and S. Y. Pi, Phys. Rev. **D37**, 3557 (1988).

[3] D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. **D49**, 2769 (1994).

[4] S. Y. Pi and M. Samiullah, Phys. Rev. **D36**, 3128 (1987).

[5] F. Cooper and E. Mottola, Phys. Rev. **D36**, 3138 (1987).

[6] J. P. Blaizot and A. Krzywicki, Phys. Rev. **D46**, 246 (1992).

[7] K. Rajagopal and F. Wilczek, Nuc. Phys. **B404**, 577 (1993).

[8] D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. **D51**, 734 (1995).

[9] D. Boyanovsky, H. J. de Vega, R. Holman, D. S. Lee and A. Singh, Phys. Rev. **D51**, 4419 (1995).

[10] A. A. Anselm, M. G. Ryskin and A. G. Shuvaev, Z. Phys. **A354**, 333 (1996); A. A. Anselm, M. G. Ryskin, Phys. Lett. **B266**, 482 (1991).

[11] J. Bjorken, Phys. Rev. **D27**, 865 (1983).

[12] F. Cooper, Y. Kluger, E. Mottola and J. Paz, Phys. Rev. **D51**, 2377 (1995).

[13] A. K. Kerman, C. Martin and D. Vautherin, Phys. Rev. **D47**, 632 (1993).

[14] A. K. Kerman and C. Y. Lin, Ann. Phys. **241**, 185 (1995).

[15] J. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10**, 2428 (1974).

[16] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), p360, Eq. (9.1.23).